LEXICOGRAPHIC GROBNER BASES OF TORIC IDEALS ARISING FROM ROOT SYSTEMS (Algorithms in Algebraic Systems and Computation Theory)

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LEXICOGRAPHIC GRÖBNER BASES OF TORIC IDEALS ARISING FROM ROOT SYSTEMS

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ABSTRACT. The present paper is a brief draft based on a joint work with Takayuki Hibi. Gröbner bases of toric ideals arising from root systems are studied.

INTRODUCTION

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set and let $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ denote the Laurent polynomial ring over a field K. We associate each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with the monomial $\mathbf{t}^{\alpha}s = t_1^{\alpha_1} \cdots t_n^{\alpha_n} s \in K[\mathbf{t}, \mathbf{t}^{-1}, s]$ and write $\mathcal{R}_K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by all monomials $\mathbf{t}^{\alpha}s$ with $\alpha \in \mathcal{A}$. Let $K[\mathbf{x}] = K[\{x_{\alpha}\, ;\, \alpha \in \mathcal{A}\}]$ denote the polynomial ring in $\sharp(\mathcal{A})$ variables over K and $I_{\mathcal{A}} \subset K[\mathbf{x}]$ the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to \mathcal{R}_K[\mathcal{A}]$ defined by setting $\pi(x_{\alpha}) = \mathbf{t}^{\alpha}s$ for all $\alpha \in \mathcal{A}$. The ideal $I_{\mathcal{A}}$ is called the *toric ideal* of the configuration $\mathcal{A}$. It is known [9] that if $I_{\mathcal{A}}$ possesses a squarefree initial ideal, then the convex hull of $\mathcal{A}$ possesses a unimodular triangulation.

Fix $n \geq 2$. Let $\mathbf{e}_i$ denote the i-th unit coordinate vector of $\mathbb{R}^n$. We write $\mathbf{A}_{n-1}^{+}$, $\mathbf{B}_n^{+}$, $\mathbf{C}_n^{+}$, $\mathbf{D}_n^{+}$ and $\mathbf{BC}_n^{+}$ for the set of positive roots of root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$, respectively ([3, pp. 64 – 65]):

$$
\begin{aligned}
\mathbf{A}_{n-1}^{+} &= \{\mathbf{e}_i - \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\};\\
\mathbf{B}_n^{+} &= \{\mathbf{e}_i\, ;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\};\\
\mathbf{C}_n^{+} &= \{2\mathbf{e}_i\, ;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\};\\
\mathbf{D}_n^{+} &= \{\mathbf{e}_i + \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j\, ;\, 1 \leq i < j \leq n\};\\
\mathbf{BC}_n^{+} &= \mathbf{B}_n^{+} \cup \mathbf{C}_n^{+}.
\end{aligned}
$$

Let, in addition, $\widetilde{\Phi}^{+} = \Phi^{+} \cup \{(0, 0, \ldots, 0)\}$, where $\Phi = \mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{BC}_n$ and where $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$.

In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [2, Theorem 6.3] discovered a squarefree quadratic initial ideal of the toric ideal $I_{\widetilde{\mathbf{A}}_{n-1}^{+}}$ of $\widetilde{\mathbf{A}}_{n-1}^{+}$. Moreover, for *any* subconfiguration $\mathcal{A}$ of $\mathbf{A}_{n-1}^{+}$, the configuration $\widetilde{\mathcal{A}} = \mathcal{A} \cup (0, 0, \ldots, 0)$ possesses a regular unimodular triangulation ([7, Example 2.4 (a)]). Stanley [8, Exercise 6.31 (b), p. 234] computed the Ehrhart polynomial of the convex polytope $\mathrm{conv}(\widetilde{\mathbf{A}}_{n-1}^{+})$. Fong [1] constructed certain triangulations of the configurations $\widetilde{\mathbf{B}}_n^{+}$ $(= \mathrm{conv}(\widetilde{\mathbf{D}}_n^{+}) \cap \mathbb{Z}^n)$

and conv($\tilde{\mathbf{C}}_n^+$) $\cap \mathbb{Z}^n$ (= $\widetilde{\mathbf{BC}}_n^+$), and computes the Ehrhart polynomials of conv($\tilde{\mathbf{B}}_n^+$) and conv($\tilde{\mathbf{C}}_n^+$). The triangulations studied in [1] are, however, non-unimodular. Motivated by their results, Ohsugi–Hibi [6] showed that

Proposition 0.1. *Let $\Phi \subset \mathbb{Z}^n$ be one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. Then, there exists a reverse lexicographic order such that the initial ideal of $I_{\tilde{\Phi}^+}$ is generated by squarefree quadratic monomials.*

Moreover, Ohsugi–Hibi [5] discussed subconfigurations $\tilde{\mathcal{A}} = \mathcal{A} \cup \{(0, 0, \ldots, 0)\}$ of $\tilde{\mathbf{B}}_n^+ \cup \tilde{\mathbf{C}}_n^+$ which possesses a (regular) unimodular triangulation (i.e., $I_{\tilde{\mathcal{A}}}$ which possesses a squarefree initial ideal).

Hence, it is natural to study the same problem as above for I_{Φ^+} where $\Phi \subset \mathbb{Z}^n$ is one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. (Then, I_{Φ^+} is not generated by quadratic binomials if $n \geq 6$.)

1. Squarefree lexicographic initial ideals

Let $\Phi^+ \subset \mathbb{Z}^n$ denote one of the configurations $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$. Let $K[\mathbf{A}_{n-1}^+]$, $K[\mathbf{B}_n^+]$, $K[\mathbf{C}_n^+]$, $K[\mathbf{D}_n^+]$ and $K[\mathbf{BC}_n^+]$ denote the polynomial rings

$$\begin{aligned}
K[\mathbf{A}_{n-1}^+] &= K[\{f_{i,j}\}_{1\leq i<j\leq n}], \\
K[\mathbf{B}_n^+] &= K[\{y_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}], \\
K[\mathbf{C}_n^+] &= K[\{a_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}], \\
K[\mathbf{D}_n^+] &= K[\{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}], \\
K[\mathbf{BC}_n^+] &= K[\{a_i\}_{1\leq i\leq n} \cup \{y_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}]
\end{aligned}$$

over K. Write $\pi : K[\Phi^+] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ for the homomorphism defined by setting

$$\pi(a_i) = t_i^2 s, \quad \pi(y_i) = t_i s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

Thus the kernel of π is the toric ideal I_{Φ^+}.

First, an explicit initial ideals of $I_{\mathbf{A}_{n-1}^+}$ generated by squarefree monomials of degree ≤ 3 will be constructed. Let $<_{lex}$ be the lexicographic order induced by the ordering of variables

$$f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n},$$

and let $<_{rev}$ be the reverse lexicographic order induced by the ordering of variables

$$f_{n-1,n} > f_{n-2,n} > f_{n-2,n-1} > \cdots > f_{2,3} > f_{1,n} > \cdots > f_{1,3} > f_{1,2}.$$

Then, the reduced Gröbner basis with respect to $<_{lex}$ (and $<_{rev}$) is as follows.

Theorem 1.1 ([4]). *The set of the binomials*

$$\begin{array}{ll}
f_{i,\ell} f_{j,k} - f_{i,k} f_{j,\ell}, & i < j < k < \ell, \\
f_{i,j} f_{j,k} - f_{i,i+1} f_{i+1,k}, & i+1 < j < k, \\
f_{i,j} f_{k,k+1} f_{k+1,\ell} - f_{i,i+1} f_{i+1,j} f_{k,\ell}, & i+1 < j < k < \ell - 1,
\end{array}$$

is the reduced Gröbner basis of the toric ideal $I_{\mathbf{A}_{n-1}^+}$ with respect to both $<_{lex}$ and $<_{rev}$, where the initial monomial of each binomial is the first monomial.

Then, we can associate the initial ideal of $I_{\mathbf{A}_{n-1}^+}$ with respect to $<_{lex}$ with the regular unimodular triangulation $\Delta_{<_{lex}}$. A graph-theoretical characterization of the maximal faces of the triangulation $\Delta_{<_{lex}}$ is given in [4].

Second, we discuss the existence of squarefree initial ideals of the toric ideal I_{Φ^+} where $\Phi \subset \mathbb{Z}^n$ is one of the root systems $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. The similar argument as in [5] plays an important rôle in the proof of Theorems 1.2 and 1.4.

Let $<_{lex}^c$ be the lexicographic order induced by the ordering of variables

$$\begin{aligned} &a_1 > a_2 > \cdots > a_n \\ &> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n} \\ &> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}. \end{aligned}$$

Theorem 1.2. *The initial ideal of the toric ideal $I_{\mathbf{C}_n^+}$ with respect to $<_{lex}^c$ is generated by squarefree monomials.*

Let $<_{lex}^d$ denote the lexicographic order obtained by restricting $<_{lex}^c$ to $K[\mathbf{D}_n^+]$. By the elimination property of the lexicographic order $<_{lex}^c$, we have the following corollary from Theorem 1.2.

Corollary 1.3. *The initial ideal of the toric ideal $I_{\mathbf{D}_n^+}$ with respect to $<_{lex}^d$ is generated by squarefree monomials.*

We now consider the root systems $\mathbf{B}_n$ and $\mathbf{BC}_n$. Let $<_{lex}^{bc}$ be the lexicographic order induced by the ordering of variables

$$\begin{aligned} &a_1 > a_2 > \cdots > a_n \\ &> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n-1} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n} \\ &> y_1 > y_2 > \cdots > y_n \\ &> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n-1} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}. \end{aligned}$$

Theorem 1.4. *The initial ideal of the toric ideal $I_{\mathbf{BC}_n^+}$ with respect to $<_{lex}^{bc}$ is generated by squarefree monomials.*

Let $<_{lex}^b$ denote the lexicographic order obtained by restricting $<_{lex}^{bc}$ to $K[\mathbf{B}_n^+]$. By the elimination property of the lexicographic order $<_{lex}^{bc}$, we have the following corollary from Theorem 1.4.

Corollary 1.5. *The initial ideal of the toric ideal $I_{\mathbf{B}_n^+}$ with respect to $<_{lex}^b$ is generated by squarefree monomials.*

Remark 1.6. Let $n \geq 6$ and let Φ^+ denote one of the configurations $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$. Then I_{Φ^+} is not generated by quadratic binomials. Hence, in particular, I_{Φ^+} does not possess a quadratic Gröbner basis.

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